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A construction of higher rank chiral polytopes

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ABSTRACT

In this paper we describe a construction for chiral polytopes with preassigned regular facets. Furthermore we show that this construction implies the existence of chiral d-polytopes, for every rank $d \geq 3$.

1. Introduction

Abstract regular polytopes are combinatorial structures allowing all possible symmetries by (abstract) reflections. They have been thoroughly studied in recent years and most of the results are summarized in [7]. In particular, it is known that for any rank d there are regular polytopes with rank d, for example those corresponding to the Coxeter groups with d generators and string diagram. Furthermore, it was known that every abstract regular polytope with rank d is the facet of an abstract regular polytope with rank $d+1$ (see for example [3,9,11]).

Chiral polytopes are defined in [12] as combinatorial structures allowing all possible symmetries by (abstract) rotations, but none at all by (abstract) reflections. The first examples of such objects appeared as quotients of regular tessellations of the Euclidean plane and of the hyperbolic space (see [2,13]), with ranks 3 and 4 respectively. Infinite chiral $(d+1)$-polytopes with preassigned chiral facets were constructed in [14]. This construction requires the $(d-1)$-faces to be regular, and therefore it can only be applied at most once to any given polytope and at most once to its dual. As a consequence, the existence of infinite chiral polytopes of ranks 5 and 6 was proved. In [1] computer software was used to construct finite chiral polytopes with ranks 4 and 5. The existence of (finite) chiral polytopes with ranks 6, 7 and 8 was recently announced.

In this paper we use techniques similar to those used in [9] to construct chiral polytopes with rank $d+1$ and with facets isomorphic to some regular polytope. This is done by using the GPR graphs defined in [10]. Furthermore we show that this leads to a constructive proof of the existence of chiral d-polytopes for every rank $d \geq 3$.

2. Regular and chiral polytopes

In this section we introduce abstract regular and chiral polytopes; see [7,12] for details.

An *abstract d-polytope* (or *abstract polytope of rank d*) $\mathcal{K}$ is a partially ordered set whose elements are called *faces* and which satisfies the following properties. $\mathcal{K}$ contains a minimum face F_{-1} and a maximum face F_d, and every *flag* of $\mathcal{K}$

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(maximal totally ordered subset) contains precisely $d+2$ elements (including F_{-1} and F_d). This induces a rank function from $\mathcal{K}$ to the set $\{-1, 0, \dots, d\}$ such that $rank(F_{-1}) = -1$ and $rank(F_d) = d$. The faces of rank i are called *i-faces*, the 0-faces are called *vertices*, the 1-faces are called *edges* and the $(d-1)$-faces are called *facets*. We shall abuse notation and casually identify the *section* $G/F_{-1} := \{H \mid H \leq G\}$ with the face G itself in $\mathcal{K}$. Given a vertex v, the section $F_d/v := \{H \mid H \geq v\}$ is called the *vertex-figure* of $\mathcal{K}$ at G. For every incident faces F and G such that $rank(F) - rank(G) = 2$, there exist precisely two faces H_1 and H_2 such that $G < H_1, H_2 < F$. This property is referred to as *diamond condition*. As a consequence of the diamond condition, for any flag Φ and any $i \in \{0, \dots, d-1\}$ there exists a unique flag Φ^i that coincides with Φ in the j-face for all $j \neq i$ but has a different i-face. This flag is called the *i-adjacent flag* of Φ. Finally, $\mathcal{K}$ must be *strongly flag-connected*, meaning that for any two flags Φ, Φ' there exists a sequence of flags $\Psi_0 = \Phi, \Psi_1, \dots, \Psi_m = \Phi'$ such that $\Phi \cap \Phi' \subseteq \Psi_k$ and Ψ_{k-1} is adjacent to Ψ_k, for $k = 1, \dots, m$.

The *dual* of a polytope $\mathcal{K}$ consists of the set of faces of $\mathcal{K}$ with the order reversed.

Whenever a d-polytope $\mathcal{K}$ has the property that for $i = 1, \dots, d-1$, the (polygonal) section $G/F := \{H \mid F \leq H \leq G\}$ between an $(i-2)$-face F and an incident $(i+1)$-face G depends only on i and not on F and G, we say that $\mathcal{K}$ is *equivelar*. In this case, for any incident faces F and G of ranks $i-2$ and $i+1$, respectively, the section G/F is isomorphic to an abstract p_i-gon for a fixed number $p_i \leq \infty$. We define the *Schläfli type* of an equivelar polytope $\mathcal{K}$ as $\{p_1, \dots, p_{d-1}\}$. Throughout this paper we will encounter only polytopes with $p_i \geq 3$ for $i = 1, \dots, d-1$. Regular and chiral polytopes defined below are examples.

An *automorphism* of a polytope $\mathcal{K}$ is an order preserving permutation of its faces. The set of automorphisms of $\mathcal{K}$ forms a group denoted by $\Gamma(\mathcal{K})$.

We say that a d-polytope $\mathcal{K}$ is *regular* if $\Gamma(\mathcal{K})$ is transitive on the flags of $\mathcal{K}$. In this case, $\Gamma(\mathcal{K})$ is generated by involutions $\rho_0, \dots, \rho_{d-1}$ where ρ_i is the (necessarily unique) automorphism of $\mathcal{K}$ mapping a fixed *base flag* Φ to its i-adjacent flag Φ^i. This set of generators satisfy the relations

$$\rho_i^2 = \varepsilon,$$
$$(\rho_i\rho_j)^2 = \varepsilon \quad \text{whenever } |i-j| \geq 2,$$

where ε denotes the identity element. These generators also satisfy the intersection conditions given by

$$\langle \rho_i \mid i \in I\rangle \cap \langle \rho_i \mid i \in J\rangle = \langle \rho_i \mid i \in I \cap J\rangle, \tag{1}$$

for all $I, J \subseteq \{0, \dots, d-1\}$. The order of the element $\rho_{i-1}\rho_i$ in $\Gamma(\mathcal{K})$ is the Schläfli number p_i (we allow $p_i = \infty$).

A group together with a generating set $\{\rho_0, \dots, \rho_{d-1}\}$ is called a *string C-group* if all generators ρ_i are involutions and they satisfy the relation $(\rho_i\rho_j)^2 = \varepsilon$ for $|i-j| \geq 2$, and the intersection condition (1). The string C-groups are in a one-to-one correspondence with the automorphism groups of regular polytopes; in particular, every regular polytope can be reconstructed from its automorphism group.

The *rotation subgroup* of (the automorphism group of) a regular d-polytope $\mathcal{K}$ is defined as the subgroup of $\Gamma(\mathcal{K})$ consisting of words of even length on the generators $\rho_0, \dots, \rho_{d-1}$ and is denoted by $\Gamma^+(\mathcal{K})$. For $i = 1, \dots, d-1$ we define the *abstract rotation* σ_i to be $\rho_{i-1}\rho_i$, that is, the automorphism of $\mathcal{K}$ mapping the base flag Φ to $(\Phi^i)^{i-1}$. It is not hard to see that $\Gamma^+(\mathcal{K}) = \langle \sigma_1, \dots, \sigma_{d-1}\rangle$ and that those generators satisfy the relations

$$(\sigma_i \cdots \sigma_j)^2 = \varepsilon \tag{2}$$

for $i < j$. The order of σ_i is just the entry p_i in the Schläfli symbol.

We define *abstract half-turns* as the involutions $\tau_{i,j} := \sigma_i \cdots \sigma_j$ for $i < j$. For consistency we also define $\tau_{0,i} := \tau_{i,d} := \varepsilon$ and denote σ_i by $\tau_{i,i}$. Then the abstract rotations and half-turns satisfy the intersection condition given by

$$\langle \tau_{i,j} \mid i \leq j; i-1, j \in I\rangle \cap \langle \tau_{i,j} \mid i \leq j; i-1, j \in J\rangle = \langle \tau_{i,j} \mid i \leq j; i-1, j \in I \cap J\rangle \tag{3}$$

for $I, J \subseteq \{-1, \dots, d\}$.

For any regular polytope $\mathcal{K}$, the index of $\Gamma^+(\mathcal{K})$ on $\Gamma(\mathcal{K})$ is at most 2. Whenever $\Gamma^+(\mathcal{K})$ has index 2 in $\Gamma(\mathcal{K})$ we say that $\mathcal{K}$ is *orientably regular*; otherwise $\mathcal{K}$ is said to be *non-orientably regular*.

We say that the d-polytope $\mathcal{K}$ is *chiral* if its automorphism group induces two orbits on flags with the property that adjacent flags always belong to different orbits. It is not hard to see that although the facets and vertex-figures of a chiral d-polytope $\mathcal{K}$ can be either regular or chiral, the $(d-2)$-faces of $\mathcal{K}$ must necessarily be regular (see [12, Proposition 9]).

Some examples of orientably regular and chiral polytopes are given by quotients of the Euclidean tessellations $\{3, 6\}$, $\{4, 4\}$ and $\{6, 3\}$ of the Euclidean plane by triangles, squares and hexagons respectively. Let T denote the group of translations of the plane which fix a given tessellation. We denote by $\{p, q\}_{\overline{u}}$ the quotient of the regular tessellation $\{p, q\}$ of the plane by the subgroup $T_{\overline{u}}$ of T generated by the translational symmetries with respect to the linearly independent vectors $\overline{u}$ and $\overline{u}R$, where R stands for the rotation by $2\pi/q$. For $\{p, q\} = \{4, 4\}$ and $\{p, q\} = \{3, 6\}$ it is well known that the quotient $\{p, q\}_{\overline{u}}$ is regular whenever $\overline{u} = (a, 0)$ or $\overline{u} = (a, a)$ with the vectors taken with respect to the basis $\{e, eR\}$ where e denotes a vector with the size and direction of an edge of the tessellation. Moreover, the quotient $\{p, q\}_{\overline{u}}$ is chiral whenever it is not regular (see [2], [7, Section 1D]). The same is true for $\{p, q\} = \{6, 3\}$ if we choose e as described above for the dual tessellation $\{3, 6\}$. Note that $\{4, 4\}_{(1,a)}$, $\{3, 6\}_{(1,0)}$ and $\{6, 3\}_{(1,0)}$ fail to be strongly flag-connected for $a = 0, 1$, and thus, they are not abstract polyhedra.

The automorphism group $\Gamma(\mathcal{K})$ of a chiral polytope is generated by elements $\sigma_1, \ldots, \sigma_{d-1}$, where σ_i maps a base flag Φ to $(\Phi^i)^{i-1}$, that is, σ_i cyclically permutes the i- and $(i-1)$-faces of $\mathcal{K}$ incident with the $(i-2)$- and $(i+1)$-faces of Φ. Furthermore, the generators σ_i also satisfy (2) as well as the intersection conditions in (3). Because of the obvious similarities between the automorphism group of a chiral polytope and the rotation subgroup of a regular polytope we shall also refer to the generators σ_i of the automorphism group of a chiral polytope as *abstract rotations*, to the products $\tau_{i,j} := \sigma_i\sigma_{i+1}\cdots\sigma_j$ $(i < j)$ as *abstract half-turns*, and to the automorphism group of a chiral polytope as its *rotation subgroup*. When convenient, we shall denote the automorphism group $\Gamma(\mathcal{K}) = \langle\sigma_1, \ldots, \sigma_{d-1}\rangle$ of the chiral polytope $\mathcal{K}$ by $\Gamma^+(\mathcal{K})$.

The following lemma provides an equivalence to some relations in (2).

Lemma 2.1. *Let $d \geq 4$, let Γ be a group generated by elements $\sigma_1, \ldots, \sigma_{d-1}$, and let $\tau_{d-2,d-1} := \sigma_{d-2}\sigma_{d-1}$. If the relations (2) hold for every $j < d-1$, then the set of relations $(\sigma_k \cdots \sigma_{d-1})^2 = \varepsilon$, $k = 1, \ldots, d-2$ is equivalent to the set of relations*

$$\tau^2_{d-2,d-1} = \varepsilon \tag{4}$$

$$\tau_{d-2,d-1}\sigma_{d-3}\tau_{d-2,d-1} = \sigma^{-1}_{d-3} \tag{5}$$

$$\tau_{d-2,d-1}\sigma_{d-4}\tau_{d-2,d-1} = \sigma_{d-4}\sigma^2_{d-3} \tag{6}$$

$$\tau_{d-2,d-1}\sigma_j = \sigma_j\tau_{d-2,d-1} \quad \textit{for } j < d-4. \tag{7}$$

Proof. The relation (4) is equivalent to $(\sigma_{d-2}\sigma_{d-1})^2 = \varepsilon$, and (5) is equivalent to $(\sigma_{d-3}\sigma_{d-2}\sigma_{d-1})^2 = \varepsilon$. Using (4) and (5), relation (6) can be seen to be equivalent to $(\sigma_{d-4}\sigma_{d-3}\sigma_{d-2}\sigma_{d-1})^2 = \varepsilon$ in the following way:

$$\begin{aligned}\sigma_{d-2}\sigma_{d-1}\sigma_{d-4}\sigma_{d-2}\sigma_{d-1} &= \sigma_{d-2}\sigma_{d-1}\sigma_{d-4}\sigma^{-1}_{d-1}\sigma^{-1}_{d-2}\sigma^{-1}_{d-3}\sigma_{d-3}\\ &= \sigma_{d-2}\sigma_{d-1}\sigma_{d-4}\sigma_{d-3}\sigma_{d-2}\sigma_{d-1}\sigma_{d-3}\\ &= \sigma^{-1}_{d-3}\sigma^{-1}_{d-4}\sigma_{d-3} = \sigma_{d-4}\sigma^2_{d-3}.\end{aligned}$$

Finally, if we assume that relations $(\sigma_k \cdots \sigma_{d-1})^2 = \varepsilon$ hold for $k > j$, Eq. (7) can be rewritten as $\sigma_j\sigma_{d-2}\sigma_{d-1}\sigma_j^{-1}\sigma^{-1}_{d-1}\sigma^{-1}_{d-2} = \varepsilon$, but then

$$\begin{aligned}\sigma_j\sigma_{d-2}\sigma_{d-1}\sigma_j^{-1}\sigma^{-1}_{d-1}\sigma^{-1}_{d-2} &= \sigma_j(\sigma_{j+1}\cdots\sigma_{d-3})^2\sigma_{d-2}\sigma_{d-1}\sigma_j^{-1}\sigma^{-1}_{d-1}\sigma^{-1}_{d-2}\\ &= \sigma^{-1}_{d-3}\sigma^{-1}_{d-4}\cdots\sigma_j^{-1}\sigma^{-1}_{d-1}\sigma^{-1}_{d-2}\cdots\sigma_j^{-1}\sigma^{-1}_{d-1}\sigma^{-1}_{d-2}\end{aligned}$$

equals ε if and only if $(\sigma_j \cdots \sigma_{d-1})^2 = \varepsilon$. □

It was proved in [12] that any group Γ together with a generating set $\{\sigma_1, \ldots, \sigma_{d-1}\}$ satisfying (2) and the intersection conditions in (3) is the rotation subgroup of either an orientably regular polytope, or a chiral polytope. Furthermore, Γ is the rotation subgroup of a regular polytope if and only if there is a group automorphism η of Γ such that

$$\sigma_i\eta = \begin{cases}\sigma_i^{-1} & \text{for } i = 1,\\ \sigma_1^2\sigma_i & \text{for } i = 2,\\ \sigma_i & \text{for } i > 2,\end{cases} \tag{8}$$

or dually,

$$\sigma_i\eta = \begin{cases}\sigma_i^{-1} & \text{for } i = d-1,\\ \sigma_i\sigma^2_{d-1} & \text{for } i = d-2,\\ \sigma_i & \text{for } i < d-2.\end{cases} \tag{9}$$

Alternatively, we may use the following criterion to determine whether a group corresponds to a regular or chiral polytope.

Proposition 2.2. *Let $\Gamma = \langle\sigma_1, \ldots, \sigma_{d-1}\rangle$ be a group satisfying (2) and the intersection condition (3). Then Γ is the rotation subgroup of the automorphism group of an orientably regular polytope if and only if there is a group automorphism α of Γ such that*

$$\begin{aligned}&\sigma_{d-1}\alpha = \sigma^{-1}_{d-1},\\ &\sigma_{d-2}\alpha = \sigma^{-1}_{d-2},\\ &\sigma_{d-3}\alpha = \sigma_{d-3}\sigma^2_{d-2}\\ &\sigma_k\alpha = \sigma_k \quad \textit{for } k \leq d-4.\end{aligned}$$

Proof. The automorphism α is the composition $\eta\eta_1$ where η is the automorphism in (9) and η_1 denotes conjugation by σ_{d-1}^{-1}. In fact, an argument similar to the one used at the end of the proof of Lemma 2.1 will prove that for $k \le d-4$, σ_{d-1} commutes with σ_k. On the other hand,

$$\sigma_{d-1}\sigma_{d-3}\sigma_{d-1}^{-1} = \sigma_{d-1}\sigma_{d-3}\sigma_{d-2}\sigma_{d-1}\sigma_{d-2} = \sigma_{d-1}\sigma_{d-1}^{-1}\sigma_{d-2}^{-1}\sigma_{d-3}^{-1}\sigma_{d-2} = \sigma_{d-3}\sigma_{d-2}^{2}. \quad \square$$

Each chiral d-polytope $\mathcal{K}$ occurs in two *enantiomorphic forms*, in a sense in a right and left handed version which can be thought of as mirror images of each other. The group $\Gamma(\mathcal{K})$ together with the distinguished generators $\sigma_1, \dots, \sigma_{d-1}$ completely determines the polytope $\mathcal{K}$ as well as the enantiomorphic form in consideration. We refer to [13] for details.

The following criterion determines that some groups satisfy the intersection condition (3) and follows directly from [12, Lemma 10].

Lemma 2.3. *Let $n \ge 4$ and let $\Gamma = \langle \sigma_1, \dots, \sigma_{d-1}\rangle$ be a group satisfying (2). If $\langle \sigma_1, \dots, \sigma_{d-2}\rangle$ is the even subgroup of the symmetry group of a regular $(d-1)$-polytope and the intersection condition*

$$\langle \sigma_1, \dots, \sigma_{d-2}\rangle \cap \langle \sigma_k, \dots, \sigma_{d-1}\rangle = \langle \sigma_k, \dots, \sigma_{d-2}\rangle$$

holds for $k = 2, \dots, d-1$, then Γ satisfies the intersection condition.

In [7, Section 7A] the mix of two regular polytopes is defined. We extend that definition to orientably regular or chiral polytopes as follows.

Let $\Gamma^+(\mathcal{K})$ and $\Gamma^+(\mathcal{P})$ be the rotation subgroups of orientably regular or chiral d-polytopes with abstract rotations given by $\sigma_1, \dots \sigma_{d-1}$ and $\delta_1, \dots, \delta_{d-1}$. The mix $\Gamma^+(\mathcal{K})\diamond\Gamma^+(\mathcal{P})$ of the groups $\Gamma^+(\mathcal{K})$ and $\Gamma^+(\mathcal{P})$ is the subgroup $\langle (\sigma_1, \delta_1), \dots, (\sigma_{d-1}, \delta_{d-1})\rangle$ of the direct product $\Gamma^+(\mathcal{K}) \times \Gamma^+(\mathcal{P})$. Whenever $\Gamma^+(\mathcal{K})\diamond\Gamma^+(\mathcal{P})$ satisfies the intersection condition with respect to the generators $\gamma_i := (\sigma_i, \delta_i)$, the group $\Gamma^+(\mathcal{K})\diamond\Gamma^+(\mathcal{P})$ is the rotation subgroup of an orientably regular or chiral polytope $\mathcal{K}\diamond\mathcal{P}$ called the *mix* of $\mathcal{K}$ and $\mathcal{P}$. The next remark follows directly from this definition.

Remark 2.4. Let $\mathcal{K}$ and $\mathcal{P}$ be two isomorphic orientably regular or chiral polytopes, then their mix is a polytope and

$$\mathcal{K}\diamond\mathcal{P} \cong \mathcal{K} \cong \mathcal{P}.$$

For any d-polytope $\mathcal{K}$ (not necessarily regular or chiral) and $i = 1, \dots, d-1$ we define the (involutory) permutation r_i on the flags of $\mathcal{K}$ by $\Phi r_i = \Phi^i$. The subgroup $\langle r_0, \dots, r_{d-1}\rangle$ of the symmetric group on the set of flags of $\mathcal{K}$ is often referred to as the *monodromy group* of $\mathcal{K}$ (see for example [6]). Note that for any flag Φ of $\mathcal{K}$ and any automorphism $\delta \in \Gamma(\mathcal{K})$, the definition of automorphism of $\mathcal{K}$ implies that $(\Phi\delta)r_i = (\Phi r_i)\delta$. An inductive argument can be used to show that, for any word w on the generators r_i,

$$(\Phi\delta)w = (\Phi w)\delta. \tag{10}$$

The following results show a useful relation between the monodromy group and the automorphism group of a chiral polytope.

Lemma 2.5. *Let $\mathcal{K}$ be a chiral polytope with base flag Φ, automorphism group $\Gamma(\mathcal{K}) = \langle \sigma_1, \dots, \sigma_{d-1}\rangle$ and monodromy group $\langle r_0, \dots, r_{d-1}\rangle$. Let $s_i := r_{i-1}r_i$ for $i = 1, \dots, d-1$. Then $\Phi\sigma_{i_1}\cdots\sigma_{i_m} = \Phi s_{i_m}^{-1} \dots s_{i_1}^{-1}$.*

Proof. We proceed by induction over m.

For $m = 1$ the result is a direct consequence of the definition of the abstract rotations σ_i. Assume that $\Phi\sigma_{i_1}\cdots\sigma_{i_{m-1}} = \Phi s_{i_{m-1}}^{-1} \dots s_{i_1}^{-1}$. Then

$$\Phi s_{i_m}^{-1}s_{i_{m-1}}^{-1}\cdots s_{i_1}^{-1} = \Phi\sigma_{i_m}s_{i_{m-1}}^{-1}\cdots s_{i_1}^{-1}.$$

As a consequence of (10), this equals $\Phi s_{i_{m-1}}^{-1}\cdots s_{i_1}^{-1}\sigma_{i_m}$ which, by the induction hypothesis, equals $\Phi\sigma_{i_1}\cdots\sigma_{i_m}$. $\square$

Corollary 2.6. *Let $\mathcal{K}$ be a chiral polytope with automorphism group $\Gamma(\mathcal{K}) = \langle \sigma_1, \dots, \sigma_{d-1}\rangle$ and monodromy group $\langle r_1, \dots, r_{d-1}\rangle$, and let $s_i := r_{i-1}r_i$ for $i = 1, \dots, d-1$. Then $s_{i_1}\cdots s_{i_m}$ stabilizes the base flag Φ of $\mathcal{K}$ if and only $\sigma_{i_1}\cdots\sigma_{i_m} = \varepsilon$.*

Proof. This follows from Lemma 2.5 and the fact that ε is the only automorphism of $\mathcal{K}$ with fixed flags. $\square$

Proposition 2.7. *Let $\mathcal{K}$ be a chiral d-polytope, $\mathcal{O}$ be the orbit of the base flag Φ of $\mathcal{K}$ under $\Gamma(\mathcal{K})$, and let $s_1, \dots, s_{d-1}$ be as in Lemma 2.5. Then there is a group isomorphism between $\Gamma(\mathcal{K})$ and the permutation group on $\mathcal{O}$ induced by $\langle s_1, \dots, s_{d-1}\rangle$.*

Proof. Note first that (10) implies that a word on $\langle s_1, \dots s_{d-1}\rangle$ fixes Φ if and only if it fixes all other flags in $\mathcal{O}$. The proposition now follows directly from Lemma 2.5 and Corollary 2.6. $\square$

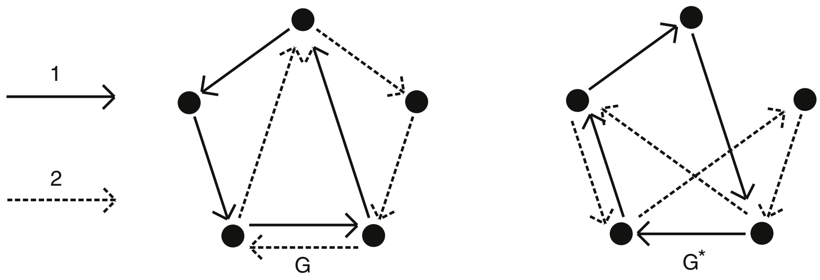

Fig. 1. GPR graph of $\{4,4\}_{(5,0)} = \overline{\{4,4\}_{(2,1)}}$.

Let $\mathcal{K}$ be a d-polytope and $\mathcal{P}$ be a regular d-polytope with automorphism group $\Gamma(\mathcal{P}) = \langle \rho_0, \ldots, \rho_{d-1} \rangle$. We say that $\mathcal{K}$ admits a *flag action* from $\mathcal{P}$ if there is a group homomorphism from $\Gamma(\mathcal{P})$ to the monodromy group of $\mathcal{K}$ mapping ρ_i to r_i. It is not hard to see that $\mathcal{K} \cong \mathcal{P}/N$ where N is the stabilizer in $\Gamma(\mathcal{P})$ of a flag of $\mathcal{K}$. The *quotient* $\mathcal{P}/N$ can be thought of as the combinatorial structure resulting from identifying the flag Φw of $\mathcal{P}$ with all flags $\Phi n w$, for Φ the base flag of $\mathcal{P}$, $n \in N$ and w in the monodromy group of $\mathcal{P}$. We refer to [5] and [7, Section 2D] for further details.

We note that whenever $\mathcal{K}$ is chiral, if the action of a word w in the generators ρ_i of $\Gamma(\mathcal{P})$ stabilizes a flag Φ of $\mathcal{K}$ then w must preserve the two orbits of flags under $\Gamma(\mathcal{K})$. Since the action of each generator ρ_i interchanges the orbits, it follows that all elements of $\Gamma(\mathcal{P})$ fixing Φ have even length, and thus, $N \subseteq \Gamma^+(\mathcal{P})$.

3. GPR graphs

CPR (C-group permutation representation) graphs are introduced in [8] as graphs encoding all information of the automorphism group of a regular polytope. This concept is extended in [10] to GPR (general permutation representation) graphs, which encode all information of the rotation subgroup of orientably regular or chiral polytopes. For convenience we include the main definition in the finite case along with some results.

Let $\mathcal{K}$ be a finite regular or chiral polytope with $\Gamma^+(\mathcal{K}) = \langle \sigma_1, \ldots, \sigma_{d-1} \rangle$ and let π be an embedding of $\Gamma^+(\mathcal{K})$ into a symmetric group S_t. The *GPR graph* of $\mathcal{K}$ determined by π is the directed multigraph (parallel arrows are allowed) with vertex set $\{1, \ldots, t\}$ and an arrow labeled $k \in \{1, \ldots, d-1\}$ from u to v if and only if $v = u(\sigma_k \pi)$.

For example, the graph at the left of Fig. 1 is the GPR graph of the chiral polyhedron $\{4,4\}_{(2,1)}$ determined by the embedding of its automorphism group on the permutations of its faces (or vertices).

GPR graphs of an orientably regular or chiral polytope $\mathcal{K}$ are in close connection with Schreier's coset diagrams of $\Gamma(\mathcal{K})$ as defined in [2, Section 3.7] (see [10, Proposition 3.4]). Each Schreier's coset diagram which is not a GPR graph corresponds to the non-faithful action of $\Gamma(\mathcal{K})$ on the cosets of some subgroup. On the other hand, all GPR graphs that are not Schreier's coset diagrams are disconnected GPR graphs.

We shall denote the arrows labeled k as *k-arrows*. For simplicity we omit all loops, and therefore, the fixed points by the image under π of σ_i are determined by the vertices contained in no k-arrow. Note that for every $k \in \{1, \ldots, d-1\}$ and every vertex v of a GPR graph there are either no k-arrows incident with v or precisely one k-arrow arriving at v and one k-arrow leaving from v. Consequently, the k-arrows form directed cycles implying that a GPR graph is a strongly connected digraph if and only if the underlying graph is connected. When convenient we shall refer to the connected components induced by arrows labeled i and by arrows labeled $i, i+1, \ldots, j$ as *i-components* and *(i, j)-components* respectively.

Whenever the embedding π plays no role or is clear from the context we shall refer to the GPR graph of $\mathcal{K}$ determined by π simply by a GPR graph of $\mathcal{K}$.

Whenever $\mathcal{K}$ is orientably regular or chiral, t is half the number of flags of $\mathcal{K}$, and π is the natural embedding of $\Gamma^+(\mathcal{K})$ into the symmetric group on the orbit of flags of $\mathcal{K}$ under $\Gamma^+(\mathcal{K})$ that contains the base flag Φ, we say that the GPR graph of $\mathcal{K}$ determined by π is the *Cayley GPR graph* of $\mathcal{K}$ and denote it by $Cay(\mathcal{K})$. Note that $Cay(\mathcal{K})$ is the (colored) Cayley graph of $\Gamma^+(\mathcal{K})$ with generators $\sigma_1, \ldots, \sigma_{d-1}$ in the sense of [15]. The following proposition and corollary follow from the regular action (transitive action with trivial point stabilizers) of $\Gamma^+(\mathcal{K})$ on one of the two induced orbits of flags of $\mathcal{K}$, or alternatively, from the regular action of a group on any of its Cayley graphs.

Proposition 3.1. *Let G be the Cayley GPR graph of an orientably regular or chiral d-polytope $\mathcal{K}$ and let u, v be two vertices of G. Then there exists a unique element in $\alpha \in \Gamma(\mathcal{K})$ such that $u\alpha = v$.*

Corollary 3.2. *Let G be the Cayley GPR graph of an orientably regular or chiral d-polytope $\mathcal{K}$. Then ε is the only element in $\Gamma(\mathcal{K})$ whose action on the vertices of G has fixed points.*

Also as a consequence of (8) and of the regular action of $\Gamma^+(\mathcal{K})$ on the vertices of its Cayley GPR graph we imply the following proposition.

Proposition 3.3. *Let $d \geq 3$ and let G be the Cayley GPR graph of an orientably regular or chiral d-polytope $\mathcal{K}$. Let G^* be the graph obtained from G by reversing all 1-arrows, erasing the 2-arrows and adding 2-arrows from v to $v\sigma_1^2\sigma_2$ for every vertex v. Then G is isomorphic to G^* if and only if $\mathcal{K}$ is regular.*

In [10, Figure 7] we include an example that shows that Proposition 3.3 is not necessarily true if we let G be any GPR graph of $\mathcal{K}$.

The group $\Gamma^+(\mathcal{K})$ acts naturally on the vertex set of the GPR graph of $\mathcal{K}$ determined by π. This action is defined by $v\sigma_i := v(\sigma_i\pi)$ and, since π is an embedding, the action is faithful. In other words, any GPR graph of a regular or chiral polytope $\mathcal{K}$ completely determines the rotation subgroup of $\mathcal{K}$.

The following result determines sets X_i of flags of the polytope $\mathcal{K}$ such that a given connected component of a GPR graph of $\mathcal{K}$ is a permutation representation graph (not necessarily faithful) on the set $X := \{X_i\}$. As a consequence, it describes the embedding π for all connected GPR graphs (cf. [10, Corollary 3.6]).

Lemma 3.4. *Let C be a connected component of a GPR graph G of an orientably regular or chiral polytope $\mathcal{K}$ with base flag Φ. Let v be a vertex of C and let Λ be the stabilizer of v under the action of $\Gamma^+(\mathcal{K})$ on $V(G)$. Then C is the GPR graph of $\mathcal{K}$ (not necessarily faithful) induced by the natural homomorphism π of $\Gamma^+(\mathcal{K})$ into the symmetric group on the set $X := \{\Phi\Lambda\gamma \mid \gamma \in \Gamma^+(\mathcal{K})\}$.*

Proof. We associate the vertex v with $\Phi\Lambda$, and the vertex $v\gamma$ with $\Phi\Lambda\gamma$ for every $\gamma \in \Gamma^+(\mathcal{K})$. The natural action of σ_i on the set X is defined by

$$(\Phi\Lambda\gamma)\sigma_i := \Phi\Lambda(\gamma\sigma_i).$$

Clearly this action can be extended to an action from $\Gamma^+(\mathcal{K})$ on X. Furthermore, there is an i-arrow from $\Phi\Lambda\gamma_1$ to $\Phi\Lambda\gamma_2$ if and only if $\Lambda\gamma_1\sigma_1 = \Lambda\gamma_2$ and, by the definition of GPR graphs, this is equivalent to having an arrow between $v\gamma_1$ to $v\gamma_2$. $\square$

The following proposition relates the automorphism group of an orientably regular or chiral polytope $\mathcal{K}$ with the automorphism group as a labeled graph of a given GPR graph of $\mathcal{K}$. The proof is analogous to that of [8, Proposition 3.7].

Proposition 3.5. *Let G be a GPR graph of an orientably regular or chiral polytope $\mathcal{K}$. Assume that Λ is the automorphism group of G as a labeled graph, and for any vertex v of G let O_v be the orbit of v under Λ. We define the quotient graph G/Λ by*

- *$V(G/\Lambda) = \{O_v \mid v \in V(G)\}$, and*
- *there is an arrow with label i from O_v to O_w if and only if there exist $x \in O_v$ and $y \in O_w$ such that there is an arrow with label i from x to y in G.*

If G/Λ is again a GPR graph of a polytope $\mathcal{P}$, then $\mathcal{P}$ is the quotient of $\mathcal{K}$ determined by the normal subgroup

$$N = \{\phi \in \Gamma^+(\mathcal{K}) \mid v\phi \in O_v \text{ for every } v \in V(G)\}.$$

Note that two distinct polytopes can have the same rotation subgroup, for instance, the cube and the hemi-cube. Consequently, any GPR graph of the cube is also a GPR graph of the hemi-cube. However, two distinct orientably regular polytopes must have distinct rotation subgroups. Therefore a GPR graph determines completely an orientably regular or chiral polytope.

The following proposition follows directly from the definition of mix of polytopes.

Proposition 3.6. *Let G and H be GPR graphs of the polytopes $\mathcal{P}$ and $\mathcal{Q}$. If the group associated with the disjoint union of G and H satisfies the intersection condition, then it is a GPR graph of the mix $\mathcal{P}\diamond\mathcal{Q}$.*

In order to construct orientably regular or chiral polytopes from GPR graphs we require to determine whether the group induced by a given graph satisfies

- relations (2) and
- the intersection condition.

Furthermore, if the group satisfies these two properties, we are usually interested in determine whether

- the induced polytope is orientably regular or chiral.

In general we cannot determine immediately from the graph whether these properties hold. In [10] we give some criteria that help in particular cases. The following theorem provides another criterion to determine that the induced group satisfies the intersection condition.

Theorem 3.7. *Let G be a graph with arrows labeled $1, \ldots, d$ and let $C_1, \ldots, C_s$ be the $(1, d-1)$-components of G. Assume also that*

(a) *$C_1, \ldots, C_s$ are isomorphic to the Cayley GPR graph of a fixed orientably regular or chiral d-polytope $\mathcal{K}$,*

(b) *for* $k = 1, \ldots, d-1$, *the action of* $(\sigma_k \cdots \sigma_d)^2$ *on the vertex set of* G *is trivial, where* σ_i *is the permutation determined by all arrows of label* i,
(c) $\langle \sigma_1, \ldots, \sigma_{d-1} \rangle \cap \langle \sigma_d \rangle = \{\varepsilon\}$.
(d) *for every* $k = 2, \ldots, d-1$ *there exist a* $(1, d-1)$*-component* C_k *and a* (k, d)*-component* D_k *such that* $C_k \cap D_k$ *is a non-empty* $(k, d-1)$*-component.*

Then G *is a GPR graph of an orientably regular or chiral polytope.*

Proof. Relations (2) follow from (a), (b) and the faithful action of $\langle \sigma_1, \ldots, \sigma_{d-1} \rangle$ on each C_i. Hence we only need to prove the intersection condition.

It follows from Remark 2.4 and (a) that $\langle \sigma_1, \ldots, \sigma_{d-1} \rangle$ is isomorphic to $\Gamma^+(\mathcal{K})$. By Lemma 2.3, it only remains to be proved that, for $2 \leq k \leq d$,

$$\langle \sigma_1, \ldots, \sigma_{d-1} \rangle \cap \langle \sigma_k, \ldots, \sigma_d \rangle = \langle \sigma_k, \ldots, \sigma_{d-1} \rangle.$$

The case $k = d$ is implied by (c). Let $2 \leq k \leq d-1$, $\alpha \in \langle \sigma_1, \ldots, \sigma_{d-1} \rangle \cap \langle \sigma_k, \ldots, \sigma_d \rangle$, and v a vertex of $C_k \cap D_k$. Since $\alpha \in \langle \sigma_1, \ldots, \sigma_{d-1} \rangle$, $v\alpha$ belongs to C_k. A similar argument shows that $v\alpha$ belong to D_k. By (d) it follows that $v\alpha$ and v belong to the same $(k, d-1)$-component, namely $C_k \cap D_k$, and therefore there exists $\beta \in \langle \sigma_k, \ldots, \sigma_{d-1} \rangle$ such that $v\alpha\beta = v$. Since $\alpha\beta \in \langle \sigma_1, \ldots, \sigma_{d-1} \rangle$ it follows from Corollary 3.2 that $\alpha = \beta^{-1} \in \langle \sigma_k, \ldots, \sigma_{d-1} \rangle$. □

In Sections 5 and 6 we shall require to construct a particular matching (set of mutually disjoint edges) between two given copies of the Cayley GPR graph of the same orientably regular d-polytope $\mathcal{K}$. Let G and G' be such copies with $u \in V(G)$ and $v \in V(G')$. We may assume that u and v correspond to the base flag of $\mathcal{K}$. Let ϕ be the group isomorphism of $\Gamma^+(\mathcal{K})$ such that $\sigma_{d-1}\phi = \sigma_{d-1}^{-1}$, $\sigma_{d-2}\phi = \sigma_{d-2}\sigma_{d-1}^2$ and $\sigma_k\phi = \sigma_k$ for $k < d-2$. We now define M as the set of edges

$$\{xy \mid x = u\xi \in V(G), y = v(\xi\phi), \xi \in \Gamma^+(\mathcal{K})\}.$$

Consider the elements of $\Gamma^+(\mathcal{K})$ as permutations of the vertices of $G \cup H$ and let α be the permutation of the vertices of $G \cup H$ that interchanges the end vertices of the edges in the matching M. It is easy to see that conjugating an element $\gamma \in \Gamma^+(\mathcal{K})$ by α is equivalent to applying ϕ on γ. This implies

Proposition 3.8. *Let* G, G' *be two graphs isomorphic to the Cayley GPR graph of an orientably regular* d*-polytope* $\mathcal{K}$ *with rotation subgroup* $\langle \sigma_1, \ldots \sigma_{d-1} \rangle$, *and let* $u \in V(G)$ *and* $v \in V(G')$. *Then there exists a perfect matching* M *in the (disjoint) union of* G *and* G' *such that* uv *is an edge of* M, *every edge of* M *contains a vertex of* G *and a vertex of* G', *and satisfies the property that*

$$\alpha\sigma_{d-1}\alpha = \sigma_{d-1}^{-1}$$
$$\alpha\sigma_{d-2}\alpha = \sigma_{d-2}\sigma_{d-1}^2$$
$$\alpha\sigma_k\alpha = \sigma_k^{-1} \quad \text{for } k < d-2$$

where the generators σ_i *are interpreted as permutations of the vertices and* μ *is the (involutory) permutation induced by* M.

The above construction of a matching can be modified to construct a matching in a single copy G of the Cayley GPR graph of an orientably regular polytope $\mathcal{K}$, with the property that no edge of the matching is incident to a given vertex u. With ϕ as above, the matching now consists of the edges

$$\{xy \mid x = u\xi \in V(G), y = u(\xi\phi), \xi \in \Gamma^+(\mathcal{K})\}.$$

Note that there may be more than one vertex not incident with any edge of this matching. In fact, there is one such vertex for each $\xi \in \Gamma^+(\mathcal{K})$ with the property that $\xi\phi = \xi$. Conjugation by the permutation induced by such a matching has the same effect in the arrows as in the construction above. It follows that

Proposition 3.9. *Let* G *the Cayley GPR graph of an orientably regular* d*-polytope* $\mathcal{K}$ *and let* $u \in V(G)$. *Then there exists a matching* M *in* G *such that no edge of* M *is incident to* u, *and*

$$\alpha\sigma_{d-1}\alpha = \sigma_{d-1}^{-1}$$
$$\alpha\sigma_{d-2}\alpha = \sigma_{d-2}\sigma_{d-1}^2$$
$$\alpha\sigma_k\alpha = \sigma_k^{-1} \quad \text{for } k < d-2$$

where the generators σ_i *are interpreted as permutations of the vertices and* μ *is the (involutory) permutation induced by* M.

As we shall see, sometimes it is convenient to consider alternative generating sets for the rotation subgroup $\Gamma^+(\mathcal{K}) = \langle \sigma_1, \ldots, \sigma_{d-1} \rangle$ of an orientably regular or chiral d-polytope $\mathcal{K}$. In Sections 5 and 6 we shall make use of $\{\sigma_1, \ldots, \sigma_{d-2}, \tau_{d-2,d-1}\}$ as generating set for $\Gamma^+(\mathcal{K})$. In doing so, we consider GPR graphs in with arrows labeled $1, \ldots, d-2$ corresponding to $\sigma_1, \ldots, \sigma_{d-2}$, and a matching corresponding to $\tau_{d-2,d-1}$ consisting of (non-oriented) edges between v and $v\tau_{d-2,d-1}$ for each vertex v. This will make it immediate to verify relation (2) for $i = d-2, j = d-1$.

4. Mixed regular cover of chiral polytopes

In the remaining sections we shall use a particular regular cover of chiral polytopes with regular facets explained next.

Let $\mathcal{K}$ be a chiral d-polytope with $\Gamma(\mathcal{K}) = \langle\sigma_1, \dots, \sigma_{d-1}\rangle$. We define the group $\overline{\Gamma(\mathcal{K})}$ as the mix $\Gamma(\mathcal{K})\diamondsuit\langle\sigma_1^{-1}, \sigma_1^2\sigma_2, \sigma_3, \dots, \sigma_{d-1}\rangle$. Clearly $\overline{\Gamma(\mathcal{K})}$ satisfies (2). Note that the group $\overline{\Gamma(\mathcal{K})}$ admits a group automorphism mapping σ_1 and σ_2 to σ_1^{-1} and $\sigma_1^2\sigma_2$ respectively while fixing σ_k for $k \geq 3$. Therefore $\overline{\mathcal{K}}$ cannot be the automorphism group of a chiral polytope. If $\overline{\Gamma(\mathcal{K})}$ is the rotation subgroup of an orientably regular polytope we call the latter the *mixed regular cover of* $\mathcal{K}$ and denote it by $\overline{\mathcal{K}}$. Note that $\gamma = (\gamma_1, \gamma_2) \in \overline{\Gamma(\mathcal{K})}$ for $\gamma_1, \gamma_2 \in \Gamma(\mathcal{K})$ if and only if γ_2 can be obtained from one word on $\sigma_1, \dots, \sigma_{d-1}$ describing γ_1 by replacing σ_1 by σ_1^{-1} and σ_2 by $\sigma_1^2\sigma_2$. Consequently, the group automorphism η of $\overline{\Gamma(\mathcal{K})}$ described in (8) interchanges the two entries of all elements.

Let $\mathcal{K}$ be a chiral polytope with mixed regular cover $\overline{\mathcal{K}}$. The definition of mixed regular cover allows us to easily construct a GPR graph of $\overline{\mathcal{K}}$ by taking the disjoint union $\overline{G}$ of any GPR graph G of $\mathcal{K}$ and the graph G^* obtained from G by reversing the 1-arrows, erasing all 2-arrows and adding an arrow labeled 2 between each vertex v and $v\sigma_1^2\sigma_2$. As a consequence of the definition of G^*, the action of an element $\gamma = (\gamma_1, \gamma_2) \in \overline{\Gamma(\mathcal{K})}$ on this graph can be interpreted as the simultaneous action of γ_1 on one copy of G and of γ_2 on another copy of G, or alternatively, as the action of γ_1 on both G and G^*.

For example, the graph at the left of Fig. 1 is a GPR graph of the chiral polyhedron $\{4, 4\}_{(2,1)}$. The union of both graphs in the figure is thus a GPR graph of the mixed regular cover $\{4, 4\}_{(5,0)}$ of $\{4, 4\}_{(2,1)}$. Note that the graph in the right of Fig. 1 is isomorphic to the one in the left after reversing all arrows.

The author does not know whether $\overline{\Gamma(\mathcal{K})}$ is the rotation subgroup of an orientably regular polytope for every chiral polytope $\mathcal{K}$. This is certainly the case if $\mathcal{K}$ has regular facets.

Proposition 4.1. *Let $d \geq 4$ and $\mathcal{K}$ be a chiral d-polytope with regular facets and automorphism group $\Gamma(\mathcal{K}) = \langle\sigma_1, \dots, \sigma_{d-1}\rangle$. Then $\overline{\Gamma(\mathcal{K})} := \Gamma(\mathcal{K})\diamondsuit\langle\sigma_1^{-1}, \sigma_1^2\sigma_2, \sigma_3, \dots, \sigma_{d-1}\rangle$ is the rotation subgroup of an orientably regular polytope, namely the mixed regular cover $\overline{\mathcal{K}}$ of $\mathcal{K}$.*

Proof. Let G be the Cayley GPR graph of $\mathcal{K}$. Then the union $\overline{G}$ of G and G^* is a permutation representation graph of $\overline{\Gamma(\mathcal{K})}$. Clearly $\overline{G}$ satisfies (a) and (b) of Theorem 3.7. Item (c) follows from the intersection condition in $\Gamma(\mathcal{K})$. In fact, if

$$(\gamma_1, \gamma_2) \in \langle(\sigma_1, \sigma_1^{-1}), (\sigma_2, \sigma_1^2\sigma_2), (\sigma_3, \sigma_3), \dots, (\sigma_{d-2}, \sigma_{d-2})\rangle \cap \langle(\sigma_{d-1}, \sigma_{d-1})\rangle$$

then $\gamma_1 = \varepsilon$ because of the intersection condition in $\Gamma(\mathcal{K})$. The fact that $(\gamma_1, \gamma_2) \in \langle(\sigma_{d-1}, \sigma_{d-1})\rangle$ now implies that γ is ε. Finally, because of the intersection condition in $\Gamma(\mathcal{K})$, the $(1, d-2)$-component and the $(k, d-1)$-component of G containing the vertex associated to the base flag of $\mathcal{K}$ will satisfy the required conditions for C_k and D_k in (d). $\square$

Proposition 4.1 is also true for $d = 3$, however the proof is out of the scope of this paper. We observe that if $\mathcal{K}$ has chiral facets then the $(1, d-2)$-components of $\overline{G}$ are not the Cayley GPR graph of a polytope and therefore we cannot use Theorem 3.7.

Proposition 4.2. *Let $\mathcal{K}$ be a chiral polytope with regular facets and let $\overline{\mathcal{K}}$ be its mixed regular cover. Then the facets of $\mathcal{K}$ are isomorphic to the facets of $\overline{\mathcal{K}}$.*

Proof. This follows from Remark 2.4, Proposition 3.6 and from the construction of the graph $\overline{G}$ in the proof of Proposition 4.1. $\square$

Note that if the facets of $\mathcal{K}$ are not regular then the facets of $\overline{\mathcal{K}}$ (in case it is polytopal) are isomorphic to the mixed regular cover of the facets of $\mathcal{K}$.

We conclude this section with the following result.

Lemma 4.3. *Let $\mathcal{K} = \mathcal{P}/N$ where $\mathcal{K}$ is a chiral d-polytope, $\mathcal{P}$ is a regular d-polytope and N is the stabilizer of a flag of $\mathcal{K}$ under the flag action of $\Gamma(\mathcal{P})$. Let G be a connected GPR graph of $\mathcal{K}$ and $\overline{G} = G \cup G^*$ be the GPR graph of $\overline{\mathcal{K}}$ defined as in the proof of Proposition 4.1. Then*

$$N = \bigcap_{v \in V(G)} Stab_{\Gamma^+(\mathcal{P})}(v), \quad \text{and} \quad \rho_0 N \rho_0 = \bigcap_{v \in V(G^*)} Stab_{\Gamma^+(\mathcal{P})}(v).$$

Proof. Let $\Gamma^+(\mathcal{K}) = \langle\sigma_1, \dots, \sigma_{d-1}\rangle$, $\Gamma^+(\mathcal{P}) = \langle\sigma_1', \dots, \sigma_{d-1}'\rangle$ and let Φ be the base flag of $\mathcal{K}$. It follows from the definition of flag action and from Corollary 2.6 that $\sigma_{i_1}' \dots \sigma_{i_m}'$ fixes Φ (and all other flags in the same orbit under $\Gamma^+(\mathcal{K})$) if and only if $\sigma_{i_1} \cdots \sigma_{i_m} = \varepsilon$.

Let Λ be the stabilizer of a vertex v of G under $\Gamma^+(\mathcal{P})$. We identify the vertex $v\gamma$ with the set of flags $\Phi\Lambda\gamma$ as in Lemma 3.4. Then the above discussion implies that $N \subseteq \cap Stab_{\Gamma^+(\mathcal{P})}(v)$. On the other hand, it follows from Lemma 2.5 and the faithful action from $\Gamma^+(\mathcal{K})$ on any of its GPR graphs that ε is the only element fixing all vertices of G, implying that $N = \cap Stab_{\Gamma^+(\mathcal{P})}(v)$.

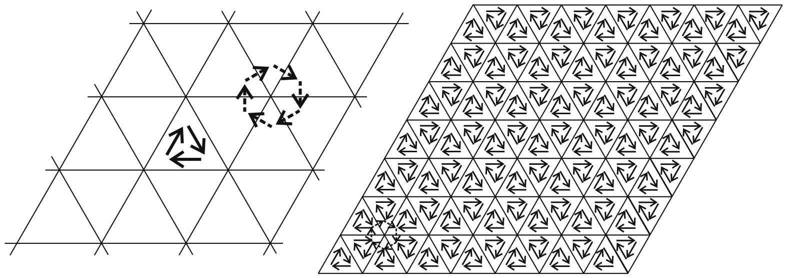

Fig. 2. Cayley GPR graph of the polyhedron $\{3,6\}_{(7,0)}$.

The equality on the right can be obtained from the equality on the left as follows. Let the vertex sets of G and G^* be $\{v_1,\ldots,v_m\}$ and $\{u_1,\ldots,u_m\}$ respectively, with the vertex u_i in G^* corresponding to the vertex v_i in G. From the construction of G^* it follows that the stabilizer of u_i in $\Gamma+(\mathcal{P})$ is $\rho_0 Stab_{\Gamma^+(\mathcal{P})}(v_i)\rho_0$, that is, the group whose elements are those of $Stab_{\Gamma^+(\mathcal{P})}(v_i)$ after replacing σ_1 and σ_2 by σ_1^{-1} and $\sigma_1^2\sigma_2$ respectively. Note that

$$\rho_0 N\rho_0 = \rho_0\left(\bigcap_{v\in V(G)} Stab_{\Gamma^+(\mathcal{P})}(v)\right)\rho_0 = \bigcap_{v\in V(G)} \rho_0 Stab_{\Gamma^+(\mathcal{P})}(v)\rho_0,$$

which concludes the proof. □

5. A chiral 4-polytope constructed from $\{3,6\}_{(2,1)}$

Before describing in detail the construction of chiral $(d+1)$-polytopes from certain chiral d-polytopes with regular facets we illustrate it with the construction of a chiral 4-polytope from the toroidal polyhedron $\{3,6\}_{(2,1)}$.

We shall construct a chiral 4-polytope $\mathcal{P}$ by describing one of its GPR graphs G. In doing so, we need to prove that the induced group satisfies (2), the intersection condition, and that there is no group automorphism of $\langle\sigma_1,\sigma_2,\sigma_3\rangle$ mapping σ_1 to σ_1^{-1} and σ_2 to $\sigma_1^2\sigma_2$ while fixing σ_3. We first explain desirable properties for G in order to satisfy these required conditions.

Theorem 3.7 suggests that we can easily prove the intersection condition if all $(1,2)$-components of G are isomorphic to $Cay(\mathcal{Q})$ for some orientably regular or chiral polyhedron $\mathcal{Q}$, which will be isomorphic to all facets of $\mathcal{P}$. Therefore we shall start our construction by considering a certain number of copies G_i of the Cayley GPR graph of an orientably regular or chiral polyhedron $\mathcal{Q}$. Furthermore, the vertex set of the graph will be given by $\cup V(G_i)$. It will remain to add all arrows corresponding to σ_3.

For simplicity, we shall construct G by using the abstract half-turn $\tau_{2,3}$ instead of σ_3, in so doing proving that $(\sigma_2\sigma_3)^2=\varepsilon$. Furthermore, Lemma 2.1 implies that $(\sigma_1\sigma_2\sigma_3)^2=\varepsilon$ is equivalent to $\tau_{2,3}\sigma_1\tau_{2,3}=\sigma_1^{-1}$. With this in mind we choose $\mathcal{Q}$ to be regular and use Propositions 3.8 and 3.9 for $d=2$ to show that the group $\langle\sigma_1,\sigma_2,\sigma_3\rangle=\langle\sigma_1,\sigma_2,\tau_{2,3}\rangle$ satisfies this relation.

To prove that the induced group is not the rotation subgroup of the automorphism group of an orientably regular polytope we shall make use of the dual version of Proposition 2.2 and prove that there is no group automorphism of $\langle\sigma_1,\sigma_2,\sigma_3\rangle$ mapping σ_1, σ_2 and σ_3 to σ_1^{-1}, σ_2^{-1} and $\sigma_2^2\sigma_3$ respectively. To this end we make use of the group structure of $\Gamma^+(\mathcal{Q})=\langle\sigma_1,\sigma_2\rangle$. Our approach requires $\Gamma^+(\mathcal{Q})$ to contain an element n with the property that n and $\rho_1 n\rho_1$ map the vertex v_1 of G (corresponding to the base flag of $\mathcal{Q}$) into two vertices in distinct 1-components of G. In this way we can manipulate the action of $n\tau_{2,3}$ and $n^{-1}\tau_{2,3}$ by choosing $\tau_{2,3}$ in a suitable way described below. For this purpose we choose $\mathcal{Q}$ to be the mixed regular cover of a chiral polyhedron $\mathcal{R}\cong\mathcal{Q}/N$ and select the element n appropriately in the subgroup N.

We now proceed to the construction. Let $\mathcal{R}=\{3,6\}_{(2,1)}$, $\mathcal{Q}=\overline{\mathcal{R}}=\{3,6\}_{(7,0)}$ and $n=(\sigma_1^{-1}\sigma_2^2)^2\sigma_1\sigma_2^{-2}$. Then, as a consequence of Proposition 2.7, the Cayley GPR graph G of $\mathcal{Q}$ is isomorphic to the graph described next. In the left hand side of Fig. 2 we show a sample i-component for $i=1,2$. The solid arrows correspond to σ_1 and are labeled 1, and the dashed arrows correspond to σ_2 and are labeled 2. The remaining connected components are obtained from these by taking their images under the translation subgroup of the toroidal polyhedron $\{3,6\}_{(7,0)}$ in thin lines in the right hand side of Fig. 2. All 1-arrows and one 2-component of the Cayley GPR graph $Cay(\mathcal{Q})$ of $\mathcal{Q}$ are shown in the right hand side of this figure.

Let $n^*=\rho_1 n\rho_1$ (as an element in $\Gamma(\{3,6\}_{7,0})$), that is,

$$n^*=(\sigma_1\sigma_2^{-2})^2\sigma_1^{-1}\sigma_2^2.$$

Note that $N=\langle n\rangle$ and that for any vertex v in the $Cay(\mathcal{Q})$, vn and vn^* belong to distinct 1-components. We shall abuse notation and denote respectively by n and n^* the elements $(\sigma_1^{-1}\sigma_2^2)^2\sigma_1\sigma_2^{-2}$ and $(\sigma_1\sigma_2^{-2})^2\sigma_1^{-1}\sigma_2^2$ in the group $\Gamma^+(\mathcal{P})$ to be constructed.

In order to construct the desired graph we follow four steps.

Step 1

We take a number $2q$ of copies of $Cay(\mathcal{Q})$, with the j-th copy having vertices $v_{i,j}$. It remains to determine the number q and the edges corresponding to $\tau_{2,3}$, which will be labeled 3.

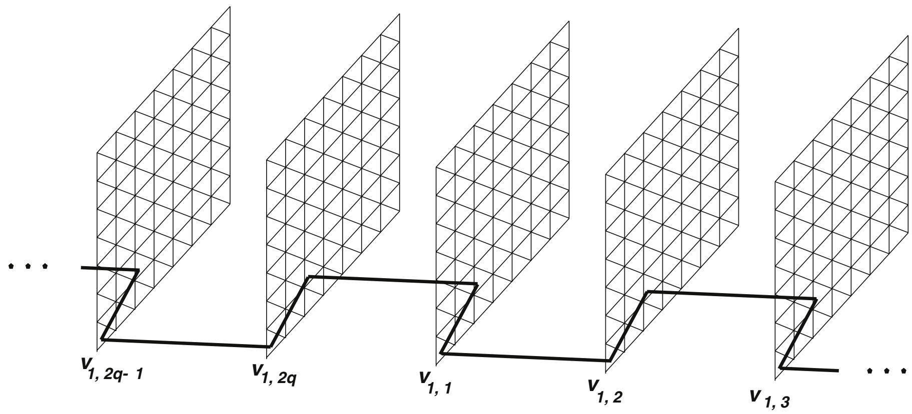

Fig. 3. Path described by $v_{1,i}$ under $(n\tau_{2,3}n^{-1}\tau_{2,3})^k$.

Step 2

We define some 3-edges in such a way that the order of

$$\omega := \tau_{2,3}n\tau_{2,3}n^{-1}$$

is a multiple of q.

Let $v_{1,j}$ denote the vertex at the left bottom corner (say) of the copy j of $Cay(\mathcal{Q})$. We add a 3-edge (corresponding to $\tau_{2,3}$) between $v_{1,j}$ and $v_{1,j+1}$ for j even, and between $v_{1,j}n$ and $v_{1,j+1}n$ for j odd (the second subindex is taken modulo $2q$). Then $v_{1,j}\omega = v_{1,j+2}$ for j odd, and $v_{1,j}\omega = v_{1,j-2}$ for j even, as illustrated in Fig. 3. Since the least positive power of ω fixing $v_{1,1}$ is q it follows that q must divide the order of ω.

Proposition 3.8 with $\Gamma(\mathcal{K}) = \langle\sigma_1\rangle$ together with each 3-edge $v_{1,j}v_{1,j+1}$ (j even) induces two other edges with the same label, namely the edge between $v_{1,j}\sigma_1$ and $v_{1,j+1}\sigma_1^{-1}$, and the edge between $v_{1,j}\sigma_1^{-1}$ and $v_{1,j+1}\sigma_1$. Similarly, each 3-edge between $v_{1,j}n$ and $v_{1,j+1}n$ (j odd) induces two other 3-edges. The 3-edges constructed so far can be interpreted in Fig. 3 as follows. In every pair of triangles of arrows labeled 1 which are joined by an edge labeled 3 and belong to consecutive copies of $Cay(\mathcal{Q})$, we add two edges to obtain a perfect matching between the two triangles in the way described in Proposition 3.8 (where G and G' are the triangles).

Step 3

We determine the remaining 3-edges in such a way that the order of

$$\omega^* := \tau_{2,3}n^*\tau_{2,3}(n^*)^{-1}$$

does not depend on q. To do this we shall define the remaining 3-edges within each copy of $Cay(\mathcal{Q})$, that is, they will join pairs of vertices which belong to the same $(1, 2)$-component. Furthermore, we may define the 3-edges simultaneously in all components in such a way that there is a 3-edge from $v_{i,j}$ to $v_{k,j}$ if and only if there is a 3-edge from $v_{i,l}$ to $v_{k,l}$ for every j, l.

One way to guarantee that the order of ω^* does not depend on q is to require that, if $v_{i,j}\omega^* = v_{k,l}$ for some i, k and $j \neq l$, then $v_{k,l}\omega^* = v_{i,j}$. That is, $(\omega^*)^2$ acts like ε in all vertices v with the property that v and $v\omega^*$ belong to distinct $(1, 2)$-components. In this way the orbit of any vertex under ω^* is either totally contained in a copy of $Cay(\mathcal{Q})$, or consists of only two elements. With this in mind we let the vertices $v_{1,j}\langle\sigma_1\rangle n^*$, $v_{1,j}n\langle\sigma_1\rangle n^*$, $v_{1,j}\langle\sigma_1\rangle(n^*)^{-1}$ and $v_{1,j}n\langle\sigma_1\rangle(n^*)^{-1}$ be fixed under the action of $\tau_{2,3}$, that is, there is no 3-edge incident to those vertices. As a consequence of this, $(\omega^*)^2$ fixes every vertex $v_{i,j}$ such that, for some $k \in \{0, 1, 2, 3\}$,

$$\begin{aligned} &v_{i,j} = v_{1,j}\sigma_1^k, \\ &v_{i,j}n^* = v_{1,j}\sigma_1^k, \\ &v_{i,j} = v_{1,j}n\sigma_1^k, \quad \text{or} \\ &v_{i,j}n^* = v_{1,j}n\sigma_1^k. \end{aligned} \tag{11}$$

Moreover, for each j, the set V_j consisting of vertices $v_{i,j}$ not mentioned in (11) is closed under the action of ω^*, and hence, the order of ω^* depends only on its order induced on each of the sets V_j, which does not depend on q. The vertices $v_{1,j}\langle\sigma_1\rangle n^*$, $v_{1,j}n\langle\sigma_1\rangle n^*$, $v_{1,j}\langle\sigma_1\rangle(n^*)^{-1}$ and $v_{1,j}n\langle\sigma_1\rangle(n^*)^{-1}$ are respectively marked with $\times$, $+$, $\diamond$ and $\square$ in Fig. 4(a). The vertex at the left bottom marked with a "$\cdot$" corresponds to $v_{1,j}$ and the other vertex marked with a "$\cdot$" corresponds to $v_{1,j}n$. Note that at most one vertex of each 1-component has been required to be fixed.

To guarantee the intersection condition in the group induced by G we only need to satisfy items (a), (b), (c) and (d) of Theorem 3.7.

Item (a) is satisfied by construction.

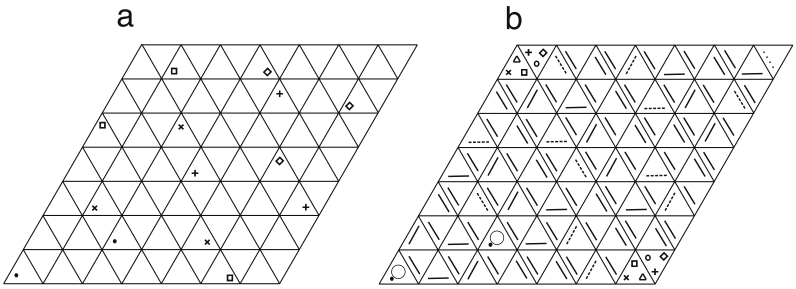

Fig. 4. Determining the edges corresponding to $\tau_{2,3}$.

To guarantee items (c) and (d) we may define a 3-edge between the vertices $v_{1,i}\sigma_2$ and $v_{1,i}\sigma_2^{-1}$ and another such edge between the vertices $v_{1,i}\sigma_2^2$ and $v_{1,i}\sigma_2^{-2}$, marked with a "+" and a "Δ" in Fig. 4(b). Additionally we require the vertex $v_{1,i}\sigma_2^3$ to be fixed under the action of $\tau_{2,3}$. In fact, the intersection of the $(1, 2)$-component and the $(2, 3)$-component containing $v_{1,i}$ is just the 2-component containing $v_{1,i}$, implying (d). On the other hand, the orbit of $v_{1,1}$ under the action of $\langle\sigma_3\rangle$ consists of the set

$$\{v_{1,1}, v_{1,1}\sigma_2, v_{1,2}, v_{1,2}\sigma_2\} \tag{12}$$

(recall that $\sigma_3 = \sigma_2^{-1}\tau_{2,3}$). Every element in $\langle\sigma_1, \sigma_2\rangle$ preserves all copies of $Cay(\mathcal{Q})$ (in particular the one containing $v_{1,1}$), however, all elements in $\langle\sigma_3\rangle$ preserving the copy $\{v_{i,1} \mid i = 1, \ldots, m\}$ act like ε in the set (12). Therefore any element in $\langle\sigma_1, \sigma_2\rangle \cap \langle\sigma_3\rangle$ must fix the vertex $v_{1,1}$, but ε is the only element in $\langle\sigma_1, \sigma_2\rangle$ with that property. This implies (c).

The edges just defined together with Propositions 3.8 and 3.9 for $d = 2$ determine the edges $\{v_{1,i}\sigma_2\sigma_1, v_{1,i}\sigma_2^{-1}\sigma_1^{-1}\}$, $\{v_{1,i}\sigma_2\sigma_1^{-1}, v_{1,i}\sigma_2^{-1}\sigma_1\}$, $\{v_{1,i}\sigma_2^2\sigma_1, v_{1,i}\sigma_2^{-2}\sigma_1^{-1}\}$ and $\{v_{1,i}\sigma_2^2\sigma_1^{-1}, v_{1,i}\sigma_2^{-2}\sigma_1\}$. The vertices contained in these edges are indicated with "$\Diamond$", "o", "$\Box$" and "$\times$" respectively in Fig. 4(b).

In order to prove item (b) of Theorem 3.7 we only need to prove that $(\sigma_1\tau_{2,3})^2 = (\sigma_1\sigma_2\sigma_3)^2 = \varepsilon$. This can be guaranteed by Proposition 3.8 (in the vertices incident with 3-edges already defined) and 3.9 if we define the remaining 3-edges in the following way. For each 1-component with no vertex incident to a 3-edge add a 3-edge between any two vertices in the connected component. Since so far we required at most one vertex in each connected component not to be contained in a 3-edge, there are always at least two vertices left to be joined by a 3-edge.

Fig. 4(b) shows the 3-edges within each copy of $Cay(\mathcal{Q})$ we consider for our example. The circles next to the "$\cdot$"s indicate vertices joined to vertices of a different copy of $Cay(\mathcal{Q})$ by a 3-edge. The dashed edges arise from the fact that the vertices $v_{1,j}\langle\sigma_1\rangle n^*$, $v_{1,j}n\langle\sigma_1\rangle n^*$, $v_{1,j}\langle\sigma_1\rangle(n^*)^{-1}$ and $v_{1,j}n\langle\sigma_1\rangle(n^*)^{-1}$ have no 3-edge incident to them (compare with Fig. 4(a)). The dotted edge at the upper right corner arises from the fact that the vertex $v_{1,1}\sigma_2^3$ is not contained in a 3-edge. The remaining edges were chosen arbitrarily. Note that there are several possible such choices.

Step 4

We determine the number q. An easy but rather tedious calculation from the graph shows that the order of ω^* is $630 = 2 \cdot 3^2 \cdot 5 \cdot 7$ regardless of our choice of q. Therefore we may choose any $q \geq 4$ which is not a divisor of 630 to guarantee that the orders of ω and ω^* are distinct. For simplicity we choose $q = 4$. It can be verified in the graph or with a computer algebra program such as GAP [4] that the order of ω is now 28.

Note that ω^* is obtained from ω by replacing σ_1, σ_2 and σ_3 by $\sigma_1^{-1}, \sigma_2^{-1}$ and $\sigma_2^2\sigma_3$. Consequently the induced group will not contain a group automorphism mapping σ_1, σ_2 and σ_3 to $\sigma_1^{-1}, \sigma_2^{-1}$ and $\sigma_2^2\sigma_3$, and thus, by the dual version of Proposition 2.2, $\Gamma(\mathcal{P})$ cannot be the rotation subgroup of a regular polytope.

The discussion above shows that the graph with 2352 vertices just constructed is a GPR graph of a chiral 4-polytope $\mathcal{P}$ with facets isomorphic to $\{3, 6\}_{(7,0)}$. It can be verified with GAP [4] that the Schläfli type is $\{3, 6, 2520\}$, however, other choices of 3-edges may yield 4-polytopes with different Schläfli types.

The sets $B_i := \{v_{i,j} \mid j \in \mathbb{Z}_8\}$ form 294 blocks of imprimitivity of the action of $\Gamma^+(\mathcal{P})$. We may identify these blocks with the vertices v_i of the Cayley GPR graph of $\{3, 6\}_{(7,0)}$ sketched in Fig. 2. Furthermore, the element

$$\phi := [(\sigma_1\sigma_3)^{120}\sigma_1(\sigma_1\sigma_3)^{120}\sigma_1^{-1}(\sigma_1\sigma_3)^{120}\sigma_1^{-1}(\sigma_1\sigma_3)^{120}\sigma_1]^2$$

induces a 3-cycle among the blocks B_i. An easy but rather tedious calculation will show that there are enough 3-cycles in the induced group to generate the alternating group on the blocks B_i corresponding to vertices on the 49 triangles facing up (say) in Fig. 2. Since σ_2 interchanges the vertices in triangles facing up with the vertices in triangles facing down, it can be proved that the induced group is $(A_{147} \times A_{147}) \rtimes C_2$. It follows that the order of $\Gamma^+(\mathcal{K})$ is divisible by $(147!)^2/2$. To determine the size of $\Gamma^+(\mathcal{P})$ it would remain to determine the elements fixing all blocks B_i, but that is out of the scope of this paper.

We note that the graph constructed in this section is a GPR graph of a chiral 4-polytope since the induced orders of ω and ω^* are 14 and 630 respectively. We chose $q = 4$ to illustrate the situation we will find in the construction described in the next section, where we will not know either the order of ω or how it is related with the order of ω^*.

6. The GPR graph $Chi_q(G)$

In the previous section we constructed a chiral 4-polytope using the structure of the chiral 3-polytope $\{3, 6\}_{(2,1)}$. Now we show one way of extending that idea to construct chiral polytopes of higher rank. We shall do this by constructing a GPR graph of them. We start by defining a particular class of chiral polytopes.

Definition 6.1. Let $d \geq 4$, let $\mathcal{K}$ be a chiral d-polytope with regular facets and let $\mathcal{Q} := \overline{\mathcal{K}}$ be its mixed regular cover defined as in Section 4 with $\Gamma(\mathcal{Q}) = \langle \rho_0, \ldots, \rho_{d-1} \rangle$ and $\Gamma^+(\mathcal{Q}) = \langle \sigma_1, \ldots, \sigma_{d-1} \rangle$. Let $H \leq \Gamma^+(\mathcal{Q})$ such that $\mathcal{K} \cong \mathcal{Q}/H$ and let $H^* = \rho_0 H \rho_0$.

We say that $\mathcal{K}$ is *scattered* if there exists an element $h \in H$ such that, for $h^* := \rho_0 h \rho_0$,

$$h, h^* \notin \langle \sigma_2, \ldots, \sigma_{d-1} \rangle \langle \sigma_1, \ldots, \sigma_{d-2} \rangle, \tag{13}$$

$$h \notin \langle \sigma_1, \ldots, \sigma_{d-2} \rangle H^* \langle \sigma_1, \ldots, \sigma_{d-2} \rangle, \tag{14}$$

and $\Gamma^+(\mathcal{Q})$ satisfies the property that its subsets

$$\langle \sigma_2, \ldots, \sigma_{d-1} \rangle \langle \sigma_1, \ldots, \sigma_{d-2} \rangle, \tag{15}$$

$$\langle \sigma_1, \ldots, \sigma_{d-2} \rangle h^* \langle \sigma_1, \ldots, \sigma_{d-2} \rangle, \tag{16}$$

$$h \langle \sigma_1, \ldots, \sigma_{d-2} \rangle h^* \langle \sigma_1, \ldots, \sigma_{d-2} \rangle, \tag{17}$$

$$\langle \sigma_1, \ldots, \sigma_{d-2} \rangle (h^*)^{-1} \langle \sigma_1, \ldots, \sigma_{d-2} \rangle, \tag{18}$$

$$h \langle \sigma_1, \ldots, \sigma_{d-2} \rangle (h^*)^{-1} \langle \sigma_1, \ldots, \sigma_{d-2} \rangle \tag{19}$$

are pairwise disjoint. Furthermore, we also require that for $\alpha, \beta \in \langle \sigma_1, \ldots, \sigma_{d-2} \rangle$,

$$\alpha h^* = h^* \beta \quad \text{implies that} \quad \alpha = \beta = \varepsilon. \tag{20}$$

We shall refer to such an h by a *scattering* element in H.

In some sense, (13) and (14) indicate that a scattered polytope contains a flag that is mapped far away from itself by the automorphism ρ_0 of $\mathcal{Q}$. The remaining properties of a scattered polytope guarantee that the polytope has enough facets to allow the construction below.

Let $\mathcal{R}$ be a scattered chiral d-polytope with $\mathcal{Q} = \overline{\mathcal{R}}$, $\mathcal{R} = \mathcal{Q}/N$ and scattering element $n \in N$. Let G be the Cayley GPR graph of $\mathcal{Q}$ with vertex set $V(G) = \{v_1, \ldots, v_r\}$ where v_1 corresponds to the base flag, $v_1, \ldots, v_s$ correspond to the flags containing the base facet F_1 of $\mathcal{Q}$, and $v_{s+1}, \ldots, v_{2s}$ are the vertices $v_1 n\alpha$ with $\alpha \in \langle \sigma_1, \ldots, \sigma_{d-2} \rangle$ and $v_{s+1} = v_1 n$. Note that $v_{s+1}, \ldots, v_{2s}$ are the vertices of a $(1, d-2)$-component of G. It follows from (13) that the sets $\{v_1, \ldots, v_s\}$ and $\{v_{s+1}, \ldots, v_{2s}\}$ are disjoint.

For each positive integer q we define the graphs $Chi_q(G)$ as follows. Let

$$V(Chi_q(G)) = \{v_{i,j} \mid i \in \{1, \ldots, r\}, j \in \mathbb{Z}_{2q}\}$$

with the following set of arrows and edges.

(I) An arrow with label $i \in \{1, \ldots, d-1\}$ from $v_{i,j}$ to $v_{k,l}$ if and only if there is an arrow in G from v_i to v_k, and $j = l$. This induces $2q$ copies of G.

(II) An edge with label d (corresponding to $\tau_{d-1,d}$) from $v_{1,j}$ to $v_{1,j+1}$ for j odd and from $v_{s+1,j}$ to $v_{s+1,j+1}$ for j even.

(III) Edges labeled d incident to the remaining vertices $v_{i,j}$ for $i \leq 2s$ determined by the edges in (II) and the matching referred in Proposition 3.8. So far $Chi_q(G)$ consists of $2q$ copies of G joined as in Fig. 5, where F_2 indicates the components consisting of the vertices $v_{i,j}$ for $s+1 \leq j \leq 2s$.

(IV) For each j, add edges labeled d between pairs of vertices in the set

$$\{v_{1,j}\alpha \mid \alpha \in \langle \sigma_2, \ldots, \sigma_{d-1} \rangle \langle \sigma_1, \ldots, \sigma_{d-2} \rangle \setminus \langle \sigma_1, \ldots, \sigma_{d-2} \rangle\} \tag{21}$$

as indicated in the following variation of the matching described in Proposition 3.9. Let ϕ be the group automorphism of $\langle \sigma_1, \ldots, \sigma_{d-1} \rangle$ fixing σ_k for each $k \leq d-4$ and mapping $\sigma_{d-3}, \sigma_{d-2}$ and σ_{d-1} to $\sigma_{d-3}\sigma_{d-2}^2, \sigma_{d-2}^{-1}$ and σ_{d-1}^{-1} respectively (see Proposition 2.2). For any element $\xi \in \langle \sigma_2, \ldots, \sigma_{d-1} \rangle \langle \sigma_1, \ldots, \sigma_{d-2} \rangle \setminus \langle \sigma_1, \ldots, \sigma_{d-2} \rangle$ add an edge labeled d between the vertices $v_{1,j}\xi$ and $v_{1,j}(\xi\phi)$. This matching is well defined in the set (21) since ϕ preserves the sets $\langle \sigma_1, \ldots, \sigma_{d-2} \rangle$ and $\langle \sigma_2, \ldots, \sigma_{d-1} \rangle$.

Note that these edges do not overlap with the edges described in (II) and (III) because of (13).

(V) For each j and for

$$v \in \{v_{1,j}\alpha n^*, v_{1,j} n\alpha n^*, v_{1,j}\alpha (n^*)^{-1}, v_{1,j} n\alpha (n^*)^{-1} \mid \alpha \in \langle \sigma_1, \ldots, \sigma_{d-2} \rangle\} \tag{22}$$

add the matching consisting of edges labeled d referred in Proposition 3.9 with $u = v$ in the $(1, d-2)$-component of v. We claim that any two vertices in the set (22) belong to distinct $(1, d-2)$-components, and thus, these edges are well defined. In fact, this follows from (20) and from the fact that the sets (16)–(19) are pairwise disjoint. In fact,

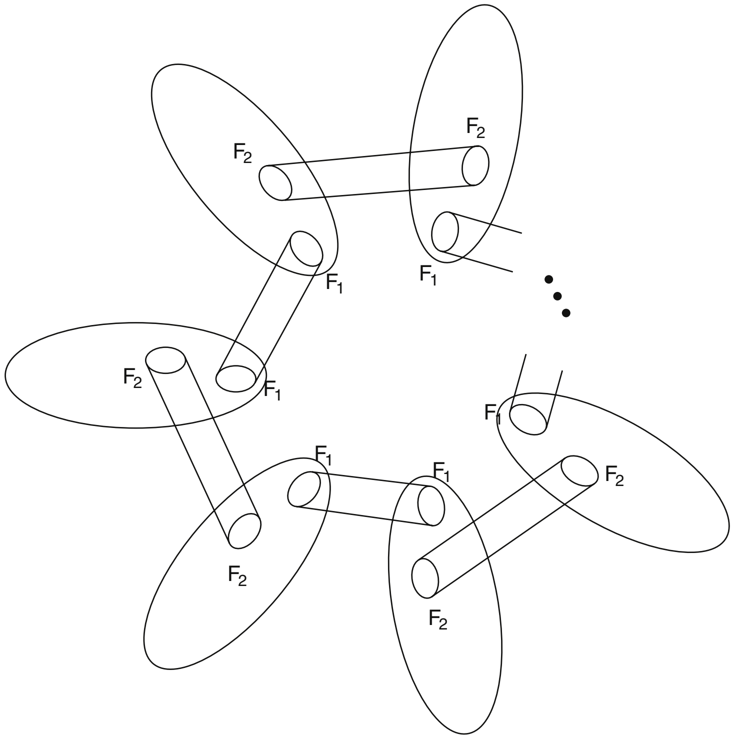

Fig. 5. Structure of the graph $Chi_q(G)$.

$v_{1,j}\alpha n^* = v_{1,j}\alpha' n^*\beta$ with $\alpha, \alpha', \beta \in \langle\sigma_1, \ldots, \sigma_{d-2}\rangle$ implies that $(\alpha')^{-1}\alpha n^* = n^*\beta$ and therefore $(\alpha')^{-1}\alpha = \beta = \varepsilon$ (similar arguments can be applied to the remaining elements in (22)).

Note that these edges do not overlap with the edges described in (IV) since the sets (16)–(19) have trivial intersection with the set (15). Furthermore, (13) and (14) imply that the edges just defined are consistent with those defined in (II) and (III).

(VI) In each $(1, d-2)$-component with vertices $v_{i,1}$ which have not been considered in (I), (II), (III), (IV) or (V), add d-edges as described in the discussion preceding Proposition 3.9 with the vertex u chosen arbitrarily within each connected component. For $j \geq 1$ add a d-edge between the vertices $v_{i,j}$ and $v_{k,j}$ if and only if there is a d-edge between the vertices $v_{i,1}$ and $v_{k,1}$.

Now we show that for almost all values of q the graph $Chi_q(G)$ is a GPR graph of a chiral $(d+1)$-polytope.

Proposition 6.2. *Let $d \geq 4$ and let $\mathcal{R}$ be a scattered chiral d-polytope with regular facets with $\mathcal{Q} = \overline{\mathcal{R}}$ the mixed regular cover of $\mathcal{R}$, $N \leq \Gamma^+(\mathcal{Q})$ such that $\mathcal{R} = \mathcal{Q}/N$ and scattering element $n \in N$.*

Let G be the Cayley GPR graph of $\mathcal{Q}$. Then there exists b such that for every $q \geq b$ the graphs $Chi_q(G)$ are GPR graphs of chiral $(d+1)$-polytopes whose facets are isomorphic to $\mathcal{Q}$.

Proof. We need to prove that the group induced by $\sigma_1, \ldots, \sigma_{d-1}, \tau_{d-1,d}$ (defined as permutations of the vertices of $Chi_q(G)$) satisfies relations (2) and the intersection condition, and that the resulting polytope is not regular. As we shall see, relations (2) and the intersection condition hold regardless of the choice of q.

For $i < j < d$ relations (2) are a consequence of the fact that every $(1, d-1)$-component is isomorphic to the Cayley GPR graph of $\mathcal{Q}$. Relations $(\sigma_i \cdots \sigma_d)^2 = \varepsilon$ follow from Lemma 2.1 and Propositions 3.8 and 3.9. Note that the set of d-edges incident to each $(1, d-1)$-component were defined from these two propositions.

The intersection condition for $\langle\sigma_1, \ldots, \sigma_d\rangle$ follows from Theorem 3.7, if we show that $Chi_q(G)$ satisfies Items (a), (b), (c) and (d). Item (a) holds by construction, Item (b) was discussed above. Item (d) follows by construction as well, taking C_k and D_k as the connected components containing $v_{1,1}$. Here we use Item (IV) of the construction of $Cay_q(G)$. Finally, Item (c) can be proved with the following argument. It follows from Item (IV) that $v_{1,j}\sigma_d = v_{1,j}\sigma_{d-1}$ for every j, and $v_{1,j}\sigma_d^2 = v_{1,j+1}$ for j odd (recall that $\sigma_d = \sigma_{d-1}^{-1}\tau_{d-1,d}$). Therefore σ_d contains the 4-cycle

$$(v_{1,1}, v_{1,1}\sigma_{d-1}, v_{1,2}, v_{1,2}\sigma_{d-1})$$

in its cyclic structure. Consequently, any power of σ_d preserving the set $\{v_{i,1} \mid i \in \{1, \ldots, r\}\}$ acts like ε in $v_{1,1}$. It follows from Corollary 3.2 that $\langle\sigma_1, \ldots, \sigma_{d-1}\rangle \cap \langle\sigma_d\rangle$ must be trivial.

To prove that for some values of q the resulting polytope is not regular we define the elements

$$\omega := \tau_{d-1,d} n \tau_{d-1,d} n^{-1}$$

and

$$\omega^* := \tau_{d-1,d} n^* \tau_{d-1,d} (n^*)^{-1}.$$

Note that ω^* can be obtained from ω by replacing σ_1 and σ_2 by σ_1^{-1} and $\sigma_1^2\sigma_2$, and therefore, it suffices to prove that ω and ω^* have distinct orders. An easy calculation will show that $(\omega^*)^2$ fixes all vertices of the form $v_{1,j}\alpha\nu$ and $v_{1,j}n\alpha\nu$ with $j \in \mathbb{Z}_{2q}$, $\alpha \in \langle \sigma_1, \ldots, \sigma_{d-1} \rangle$ and $\nu \in \{\varepsilon, (n^*)^{-1}\}$. Furthermore, ω^* preserves the sets of vertices

$$V_j := \{v_{i,j} \mid i = 1, \ldots, r\} \setminus \{v_{1,j}\alpha\nu, v_{1,j}n\alpha\nu \mid \alpha \in \langle \sigma_1, \ldots, \sigma_{d-1} \rangle, \nu \in \{\varepsilon, (n^*)^{-1}\}\}.$$

Consequently, the order of ω^* does not depend on q. On the other hand, the smallest positive power of ω fixing $v_{1,1}$ is q and therefore q must divide the order of ω. Hence, the graphs $Chi_q(G)$ are GPR graphs of chiral polytopes for every q not dividing the order of ω^*, for instance, any integer q greater than the order of ω^*.

It follows from Remark 2.4 and Proposition 3.6 that the facets of the resulting polytope are isomorphic to $\mathcal{Q}$. □

We observe that in the previous section, where we construct a chiral 4-polytope $\mathcal{P}$ from a polyhedron, the group automorphism of $\Gamma^+(\mathcal{P}) = \langle \sigma_1, \sigma_2, \sigma_3 \rangle$ mapping σ_1, σ_2 and σ_3 to σ_1^{-1}, $\sigma_1^2\sigma_2$ and σ_3 respectively maps $\tau_{2,3}$ to $\sigma_1^2\tau_{2,3}$ and therefore we need to use the dual version of Proposition 2.2 instead of (8).

The following lemma summarizes some properties of the chiral polytopes above constructed.

Lemma 6.3. *Let $\mathcal{R}$, $\mathcal{Q}$, N, G and $Chi_q(G)$ be as in Proposition 6.2. Let $\omega := \tau_{d-1,d} n \tau_{d-1,d} n^{-1}$ and $\omega^* := \tau_{d-1,d} n^* \tau_{d-1,d} (n^*)^{-1}$. Then*

(a) *any element in the group $\langle \sigma_1, \ldots, \sigma_{d-1} \rangle$ maps $v_{i,j}$ to $v_{k,j}$ for some k,*
(b) *any element of $\langle \sigma_2, \ldots, \sigma_d \rangle$ maps $v_{i,j}$ to $v_{k,l}$ for some $k \in \{1, \ldots, r\}$ and some $l \in \{j-1, j, j+1\}$,*
(c) *for any integer a, $v_{i,j}(\omega^*)^a = v_{k,l}$ for some $k \in \{1, \ldots, r\}$ and some $l \in \{j-1, j, j+1\}$, moreover, $l = j$ whenever a is even,*
(d) *$v_{i,j}\omega = v_{k,l}$ for some $k \in \{1, \ldots, r\}$ and some $l \in \{j-2, j-1, j, j+1, j+2\}$.*
(e) *$n\langle \sigma_1, \ldots, \sigma_{d-2} \rangle n \cap \langle \sigma_1, \ldots, \sigma_{d-2} \rangle = \varepsilon$.*

Proof. Item (a) follows directly from (I) in the construction of $Chi_q(G)$. Item (b) is a consequence of (13) in the polytope $\mathcal{Q}$. The proof of Item (c) is given in the proof of Proposition 6.2. Item (d) is implied by the fact that ω has only two factors $\tau_{d-1,d}$. Item (e) follows form the fact that the sets (16) and (18) (with H and h replaced by N and n respectively) are disjoint. In fact, if η is the group automorphism of $\langle \sigma_1, \ldots, \sigma_{d-2} \rangle$ mapping σ_1 and σ_2 to σ_1^{-1} and $\sigma_1^2\sigma_2$ while preserving σ_k for $k \geq 3$, and $\alpha, \beta \in \langle \sigma_1, \ldots, \sigma_{d-2} \rangle$ are such that $n\alpha = \beta n^{-1}$ then $n^*(\alpha\eta) = (\beta\eta)(n^*)^{-1}$. But $\alpha\eta, \beta\eta \in \langle \sigma_1, \ldots, \sigma_{d-2} \rangle$, which contradicts our hypothesis. □

The following lemma allows us to apply the above construction repeatedly.

Lemma 6.4. *Let $d \geq 4$, $\mathcal{R}$ be a chiral scattered $(d-1)$-polytope with regular facets and mixed regular cover $\mathcal{Q} := \overline{\mathcal{R}}$, and let G be the Cayley GPR graph of $\mathcal{Q}$.*

Let $\mathcal{P}$ be the chiral d-polytope with GPR graph given by a graph $Chi_q(G)$ for certain $q \geq 2b$ for b as in Proposition 6.2. Then $\mathcal{P}$ is scattered.

Proof. Let $\mathcal{K} := \overline{\mathcal{P}}$ be the mixed regular cover of $\mathcal{P}$ with $\Gamma(\mathcal{K}) = \langle \rho_0, \ldots, \rho_{d-1} \rangle$ and $\Gamma^+(\mathcal{K}) = \langle \sigma_1, \ldots, \sigma_{d-1} \rangle$. Let $M \leq \Gamma^+(\mathcal{K})$ such that $\mathcal{P} = \mathcal{K}/M$.

We need to show that there exists $m \in M$ such that (13), (14) and (20) hold, and the sets (15)–(19) are pairwise disjoint after replacing h and H by m and M respectively,.

Let $N \leq \mathcal{Q}$ such that $\mathcal{R} = \mathcal{Q}/N$, let $\Gamma(\mathcal{Q}) = \langle \mu_0, \ldots, \mu_{d-2} \rangle$, let $n \in N$ be the scattering element, let $n^* := \mu_0 n \mu_0$, and let $\omega := \tau_{d-2,d-1} n \tau_{d-2,d-1} n^{-1}$ and $\omega^* := \tau_{d-2,d-1} n^* \tau_{d-1,d-1} (n^*)^{-1}$. Let t be the order of ω^* in $\Gamma(\mathcal{P})$. We abuse notation and let $w := \tau_{d-2,d-1} n \tau_{d-2,d-1} n^{-1}$ and $w^* := \tau_{d-2,d-1} n^* \tau_{d-2,d-1} (n^*)^{-1}$ as elements of $\Gamma^+(\overline{\mathcal{P}})$.

Let E be the GPR graph of the rotation subgroup of $\overline{\mathcal{P}}$ obtained as the disjoint union of $Chi_q(G)$ and the graph $[Chi_q(G)]^*$. We recall that $[Chi_q(G)]^*$ is the graph obtained from $Chi_q(G)$ by reversing all 1-arrows, erasing all 2-arrows and adding a 2-arrow for each path of G determined by $\sigma_1^2\sigma_2$. For every i, j we denote by $u_{i,j}$ the vertex of $[Chi_q(G)]^*$ corresponding to $v_{i,j}$, and thus, the vertex set of E is $\{v_{i,j}, u_{i,j} \mid i = 1, \ldots, r; j \in \mathbb{Z}_{2q}\}$. Lemma 4.3 and our choice of $q \geq 2t$ imply that $(w^*)^t \in M$ and that $(w^*)^t$ (as well as the entire group M) acts like ε on all vertices $v_{i,j}$ whereas w^t (and the entire group M^*) acts like ε on all vertices of $u_{i,j}$. Note also that

$$u_{1,1}(w^*)^t = u_{1,1+2t} \quad \text{and} \quad v_{1,1}w^t = v_{1,1+2t}. \tag{23}$$

We now prove that for $m = (w^*)^t$, relations (13) and (14) after replacing h by m are satisfied. Note that (13) follows from Lemma 6.3(a) and (b) applied to $v_{1,1}$, and from (23). On the other hand, (14) is a consequence of (23), Lemma 6.3(a) and the fact that M^* acts like ε in every vertex $u_{i,j}$, in particular in $u_{1,1}$.

Next we show that (20) holds. Assume that $\alpha m^* = m^*\beta$ for some $\alpha, \beta \in \langle \sigma_1, \dots, \sigma_{d-2}\rangle$ then the regular action of $\langle \sigma_1, \dots, \sigma_{d-2}\rangle$ on the set of vertices $\{u_{i,1} \mid i \in \{1, \dots, m\}\}$ and the fact that M^* acts like ε on $u_{i,j}$ for all i, j imply that $\alpha = \beta$. Observe that $v_{1,1}m^*\beta = v_{i,1+2t}$ for some i. It follows that $(v_{1,1}\alpha)m^* = v_{i,1+2t}$. By Lemma 6.3(d) and our assumption of $q \geq 2b$, $(v_{1,1}\alpha)\omega^k = v_{i,2k+1}$ for some i, in particular $(v_{1,1}\alpha)\omega = v_{i,3}$ for some i. It follows directly from the construction that $\alpha \in \langle \sigma_1, \dots, \sigma_{d-3}\rangle$, but since $\mathcal{Q}$ satisfies (20), α must be ε.

To prove that the set (15) is disjoint to the sets (16)–(19) it suffices to make use of Lemma 6.3(a) and (b) and note that for every element ξ of each of the other four sets there is an $\alpha \in \langle \sigma_1, \dots, \sigma_{d-2}\rangle$ such that ξ maps the vertex $v_{1,1}\alpha$ or the $u_{1,1}\alpha$ into the vertex $v_{i,l}$ or the vertex $v_{i,l}$ for some $l \in \{1 + 2t, 1 - 2t\}$ and some i.

Sets (16) and (18) have empty intersection with sets (17) and (19) since the first two preserve the connected components of $[Chi_q(G)]^*$ whereas every element in the other two sets map the vertex $u_{1,1}$ to a vertex $v_{i,2t}$ for some i.

It only remains to show that the sets (16) and (18) are disjoint. Assume to the contrary that $m^*\alpha = \beta(m^*)^{-1}$ for some $\alpha, \beta \in \langle \sigma_1, \dots, \sigma_{d-2}\rangle$. Then

$$v_{1,1}\beta\omega^{-t} = v_{1,1}\beta(m^*)^{-1} = v_{1,1}m^*\alpha = v_{1,1}w^t\alpha = v_{i,1+2t}$$

for some i. It follows from Lemma 6.3(d) that $v_{1,1}\beta\omega^{-1} = v_{1,1}\beta n\tau_{d-1,d}n^{-1}\tau_{d-1,d} = v_{k,3}$ for some k. This is possible only if $\beta n \in \langle \sigma_1, \dots, \sigma_{d-3}\rangle$ and $n^{-1} \in \langle \sigma_1, \dots, \sigma_{d-3}\rangle n\langle \sigma_1, \dots \sigma_{d-3}\rangle$, since otherwise the factor $\tau_{d-1,d}$ preserves the $(1, d-3)$-component. But this contradicts Lemma 6.3(e) for $\mathcal{Q}$. Note that this immediately implies that sets (17) and (19) are disjoint as well. □

Finally we are ready to prove the main theorem.

Theorem 6.5. *For every $d \geq 3$ there exist chiral d-polytopes.*

Proof. The theorem follows from Proposition 6.2 and Lemma 6.4 provided we show a chiral scattered 4-polytope

We observe that the polytope with GPR graph isomorphic to the graph constructed as in Section 5 for $q = 1300$ is scattered. The proof of this claim is analogous to that of Lemma 6.4. □

Unfortunately the automorphism groups of the chiral polytopes constructed with this method can be extremely big, as illustrated in the previous section. This makes it hard to explore their structure. However we can give some quotient relations among them. Provided $\tau_{d,d-1}$ acts in the same way on all $(1, d-2)$-components (note that this condition can be dropped), it follows from Proposition 3.5 that, for all positive integers k and q, the group induced by the graph $Chi_q(G)$ is a quotient of the group induced by $Chi_{kq}(G)$.

In general Item (VI) of the construction of $Chi_q(G)$ can be done in several different ways. Note that the choices of d-edges have a direct impact in the last entry of the Schläfli type. For instance, if in the polytope constructed in Section 5 we replace the edge between the vertices $v_{1,1}\sigma_1^{-1}\sigma_2^2$ and $v_{1,1}\sigma_1^{-1}\sigma_2^2\sigma_1^{-1}$ (that is, the horizontal edge in the bottom row in Fig. 4) by the edge between the vertices $v_{1,1}\sigma_1^{-1}\sigma_2^2\sigma_1$ and $v_{1,1}\sigma_1^{-1}\sigma_2^2\sigma_1^{-1}$, the last entry of the Schläfli type of the chiral 4-polytope changes from 2520 to 7560. This suggests that there are many chiral $(d+1)$-polytopes that can be obtained from a chiral scattered d-polytope with the construction described above.

The requirements for a polytope to be scattered were chosen to simplify the proofs, however, they may be more restrictive than needed. For example, the polyhedron $\{4, 4\}_{(5,0)}$ is the mixed regular cover of the chiral polyhedron $\{4, 4\}_{(2,1)}$. Furthermore, $\{4, 4\}_{(2,1)} \cong \{4, 4\}_{(5,0)}/N$ where N is the cyclic group generated by the translation by the vector $(2, 1)$. Then for any $n \in N$, $\sigma_1^2 n = n^{-1}\sigma_1^2$ (this says that the sets (16) and (18) are not disjoint). If we want to construct a chiral 4-polytope in the way described in Section 5 we need to consider the fact that the vertices $v_{1,1}n$ and $v_{1,1}\sigma_2^2 n^{-1}$ belong to the same 1-component. However, it is still possible to follow Step 3 of the construction in Section 5 since there is a matching of the corresponding connected component fixing both vertices. This suggests that we can probably relax the requirements in the definition of scattered polytopes, allowing more regular polytopes to be facets of chiral polytopes obtained by this construction.

The requirement that the polytope $\mathcal{Q}$ is a regular cover of a chiral polytope is not essential. The construction of a chiral $(d+1)$-polytope from a regular d-polytope $\mathcal{Q}$ with automorphism group $\langle \rho_0, \dots, \rho_{d-1}\rangle$ and rotation subgroup $\langle \sigma_1, \dots, \sigma_{d-1}\rangle$ can be implemented if we require $\Gamma(\mathcal{Q})$ to contain an element h satisfying (13), (20) and (14) after replacing H^* by $\langle h^*\rangle$, and with the property that the sets (15)–(19) are mutually disjoint, where $h^* = \rho_0 h \rho_0$.

The last remarks suggest that not only there exist chiral d-polytopes for every rank $d \geq 3$, but that there are plenty of them.

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